1.

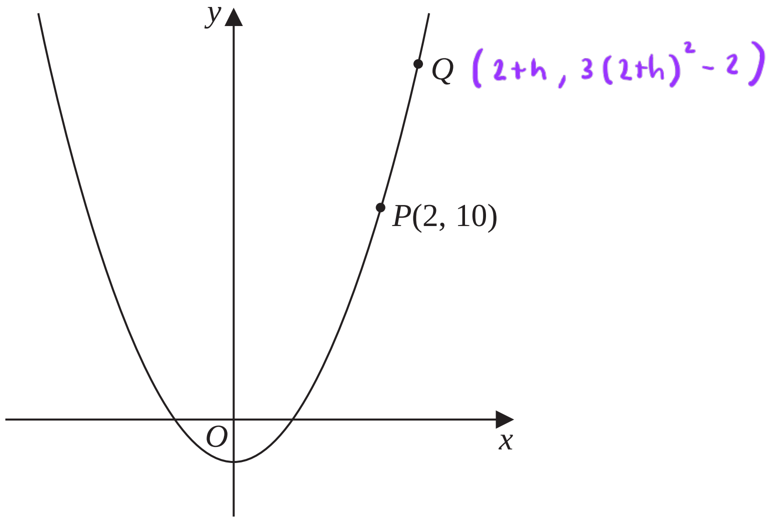


Figure 1

Figure 1 shows part of the curve with equation $y = 3x^2 - 2$

The point $P(2, 10)$ lies on the curve.

(a) Find the gradient of the tangent to the curve at P.

(2)

The point Q with x coordinate $2 + h$ also lies on the curve.

(b) Find the gradient of the line PQ, giving your answer in terms of h in simplest form.

(3)

(c) Explain briefly the relationship between part (b) and the answer to part (a).

(1)

a) $y = 3x^2 - 2$

$\Rightarrow \frac{dy}{dx} = 6x$ At P : $\frac{dy}{dx} = 6(2) = 12$ (1)

Gradient of tangent at P is 12. (1)

b) Coordinates at Q = $\left(2+h,\ 3(2+h)^2-2\right)$

$$\frac{\Delta y}{\Delta x} = \text{Gradient } PQ = \frac{3(2+h)^2-2-10}{(2+h)-2} \quad (1)$$

$$= \frac{3(2+h)^2-12}{(2+h)-2} = \frac{12h+3h^2}{h} \quad (1)$$

$$= 12+3h \quad (1)$$

c) As $h \to 0$, the gradient PQ, $12+3h \to 12$.

So, as Q gets closer to P, the gradient of the chord tends toward the instantaneous gradient of the curve at P. (1)

2.

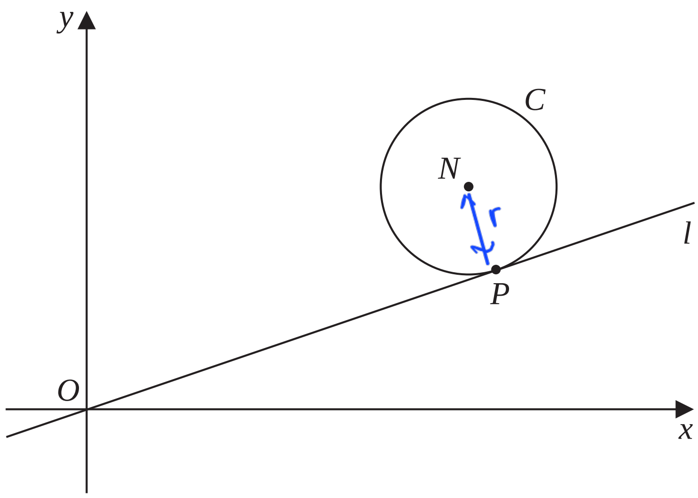


Figure 4

Figure 4 shows a sketch of a circle C with centre $N(7, 4)$

The line l with equation $y = \frac{1}{3}x$ is a tangent to C at the point P.

Find $m_l = \frac{1}{3}$

(a) the equation of line PN in the form $y = mx + c$, where m and c are constants,

(2)

(b) an equation for C.

(4)

The line with equation $y = \frac{1}{3}x + k$, where k is a non-zero constant, is also a tangent to C.

(c) Find the value of k. $m_{PN} = \frac{-1}{m_l}$

(3)

a) PN is perpendicular to l, so $m = -3$, and as N is on the line we have point $(7, 4)$

$\Rightarrow y - 4 = -3(x - 7)$ (1)

$y = -3x + 25$ (1)

b) The radius of C, r = length NP

P is the intersect of $y = \frac{1}{3}x$ and $y = -3x + 25$

At P : $\frac{1}{3}x = -3x + 25$ (1)

$x = -9x + 75$

$10x = 75 \Rightarrow x = 7.5$

$y = \frac{1}{3}(7.5) = 2.5 \quad \therefore P(7.5, 2.5)$ (1)

Length PN $= \sqrt{(7.5-7)^2 + (4-2.5)^2} = \sqrt{5/2}$ (1)

C : $(x-7)^2 + (y-4)^2 = 5/2$ (1)

c) When $y = \frac{1}{3}x + k$ satisfies the equation for C

$(x-7)^2 + \left(\frac{1}{3}x + k - 4\right)^2 = \frac{5}{2}$

$x^2 - 14x + 49 + \frac{1}{9}x^2 + \frac{k}{3}x - \frac{4}{3}x + \frac{k}{3}x + k^2 - 4k$
$- \frac{4}{3}x - 4k + 16 = \frac{5}{2}$

$\Rightarrow \frac{10}{9}x^2 + \left(\frac{2}{3}k - \frac{50}{3}\right)x + k^2 - 8k + \frac{125}{2} = 0$ (1)

This quadratic must only have one solution, as the tangent only meets the circle once.

$\Rightarrow b^2 - 4ac = 0$

$\Rightarrow \left(\frac{2}{3}k - \frac{50}{3}\right)^2 - 4\left(\frac{10}{9}\right) \times \left(k^2 - 8k + \frac{125}{2}\right) = 0$ (1)

$= \frac{4}{9}k^2 - \frac{200}{9}k + \frac{2500}{9} - \frac{40}{9}k^2 + \frac{320}{9}k - \frac{2500}{9} = 0$

$= 4k^2 - 200k - 40k^2 + 320k = 0$

$\Rightarrow -36k^2 + 120k = 0$

$k = 0$ is the case for line L.

$-36k + 120 = 0$

$k = \frac{120}{36} = \frac{10}{3}$ for the non-zero constant (1)

equation: $y = \frac{1}{3}x + \frac{10}{3}$

3. **In this question you should show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

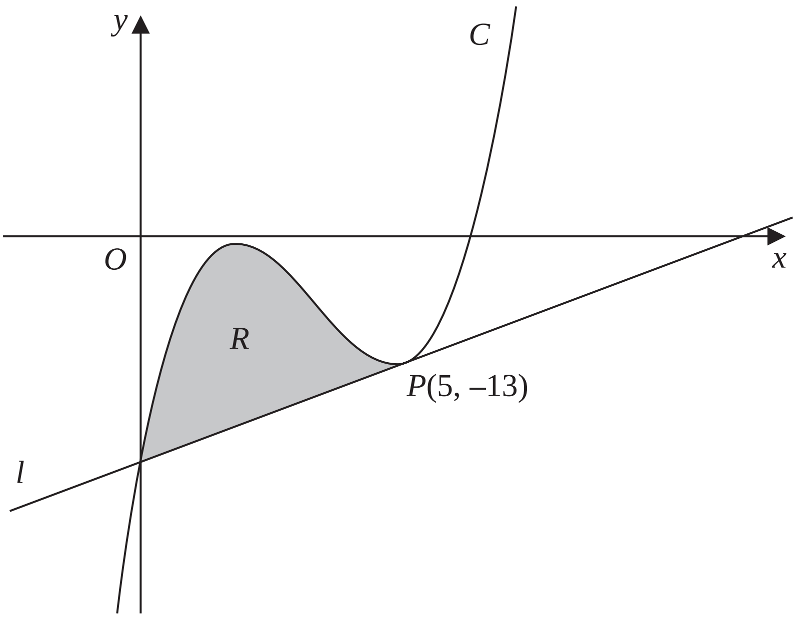


Figure 2

Figure 2 shows a sketch of part of the curve C with equation

$$y = x^3 - 10x^2 + 27x - 23$$

The point $P(5, -13)$ lies on C

The line l is the tangent to C at P

(a) Use differentiation to find the equation of l, giving your answer in the form $y = mx + c$ where m and c are integers to be found.

(4)

(b) Hence verify that l meets C again on the y-axis.

(1)

The finite region R, shown shaded in Figure 2, is bounded by the curve C and the line l.

(c) Use algebraic integration to find the exact area of R.

(4)

a) $y = x^3 - 10x^2 + 27x - 23$

$\frac{dy}{dx} = 3x^2 - 20x + 27$ (1)

$\left.\frac{dy}{dx}\right|_{x=5} = 3(5)^2 - 20(5) + 27 = 2$ (1)

$\therefore$ l has gradient 2 and goes through $(5, -13)$

$\Rightarrow$ $y + 13 = 2(x - 5)$ (1)

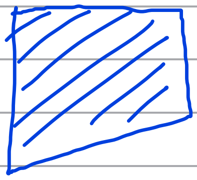
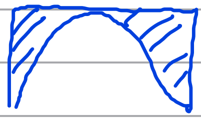
$y + 13 = 2x - 10$

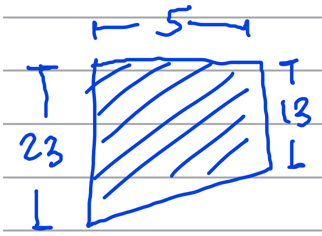
$y = 2x - 23$ (1)

b) on the y-axis, $x=0$

$C: \; y = 0^3 - 10(0)^2 + 27(0) - 23 = -23$
$L: \; y = 2 \times 0 - 23 = -23$

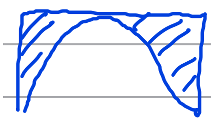
Both C and l pass through $(0, -23)$
So C meets l again on the y-axis (1)

c) Area $R =$  $-$ 



Area $= \dfrac{13+23}{2} \times 5 = 90$ (1)

- at the front since this area is below the x-axis, it will be negative. We are interested in the positive area.



$$\text{Area} = -\int_0^5 (x^3 - 10x^2 + 27x - 23)\, dx$$

$$= -\left[\frac{1}{4}x^4 - \frac{10}{3}x^3 + \frac{27}{2}x^2 - 23x\right]_0^5 \quad (1)$$

$$= \frac{455}{12} \quad (1)$$

$$R = 90 - \frac{455}{12} = \frac{625}{12} \quad (1)$$

4. The curve C has equation

$$px^3 + qxy + 3y^2 = 26$$

where p and q are constants.

(a) Show that

$$\frac{dy}{dx} = \frac{apx^2 + bqy}{qx + cy}$$

where a, b and c are integers to be found.

(4)

Given that

- the point $P(-1, -4)$ lies on C
- the normal to C at P has equation $19x + 26y + 123 = 0$

(b) find the value of p and the value of q.

(5)

a) $px^3 + qxy + 3y^2 = 26$ $\qquad \frac{d}{dx}(qxy) = qy + qx\frac{dy}{dx}$ 1

$3px^2 + qy + qx\frac{dy}{dx} + 6y\frac{dy}{dx} = 0$ 1

$3px^2 + qy + \frac{dy}{dx}(qx + 6y) = 0$

$\frac{dy}{dx}(qx + 6y) = -3px^2 - qy$ 1

$\frac{dy}{dx} = \frac{-3px^2 - qy}{qx + 6y}$ 1

b) $P(-1,-4)$ lies on C:

$$p(-1)^3 + q(-1)(-4) + 3(-4)^2 = 26 \quad 1$$
$$-p + 4q + 48 = 26$$
$$-p + 4q = -22 \quad (1)$$

Normal to C at P has equation $19x + 26y + 123 = 0$

$$\Rightarrow y = -\frac{19}{26}x - \frac{123}{26} \quad \therefore m = -\frac{19}{26} \quad 1$$

$$m_{normal} = \frac{-1}{m_{tangent}}$$

$$\left.\frac{dy}{dx}\right|_{\substack{x=-1 \\ y=-4}} = -\frac{1}{-\frac{19}{26}} = \frac{26}{19}$$

$$\Rightarrow \frac{-3p(-1)^2 - q(-4)}{q(-1) + 6(-4)} = \frac{26}{19} \quad 1$$

$$\frac{-3p + 4q}{-q - 24} = \frac{26}{19}$$

$$19(-3p + 4q) = 26(-q - 24)$$

$$-57p + 76q = -26q - 624$$

$$624 = 57p - 102q \quad (2)$$

1

solve (1) and (2) simultaneously using calculator:

$p = 2 \quad q = -5$

1

5. The curve C has parametric equations

$$x = \sin 2\theta \qquad y = \operatorname{cosec}^3 \theta \qquad 0 < \theta < \frac{\pi}{2}$$

(a) Find an expression for $\frac{dy}{dx}$ in terms of θ

(3)

(b) Hence find the exact value of the gradient of the tangent to C at the point where $y = 8$

(3)

a) $y = \operatorname{cosec}^3\theta$

$\frac{dy}{d\theta} = 3\operatorname{cosec}^2\theta \times -\operatorname{cosec}\theta \cot\theta$

$= -3\operatorname{cosec}^3\theta\cot\theta$ (1)

$\frac{dy}{dx} = \frac{dy}{d\theta} \div \frac{dx}{d\theta}$ (1)

$x = \sin 2\theta$

$\frac{dx}{d\theta} = 2\cos 2\theta$

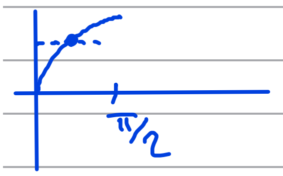
$\frac{dy}{dx} = \frac{-3\operatorname{cosec}^3\theta\cot\theta}{2\cos 2\theta}$ (1)

b) find θ when $y = 8$

$8 = \operatorname{cosec}^3\theta$

$\operatorname{cosec}\theta = 2$

$\sin\theta = \frac{1}{2}$ (1)



$0 < \theta < \frac{\pi}{2}$

$\Rightarrow \theta = \frac{\pi}{6}$

$\theta = \frac{\pi}{6} \Rightarrow \frac{dy}{dx} = \frac{-3\operatorname{cosec}^3\frac{\pi}{6}\cot\frac{\pi}{6}}{2\cos\frac{2\pi}{6}}$ (1)

$\frac{dy}{dx} = -24\sqrt{3}$ (1)

6. **In this question you must show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

The curve C has equation $y = \mathrm{f}(x)$ where $x \in \mathbb{R}$

Given that

- $\mathrm{f}'(x) = 2x + \frac{1}{2}\cos x$
- the curve has a stationary point with x coordinate α
- α is small

(a) use the small angle approximation for $\cos x$ to estimate the value of α to 3 decimal places.

(3)

The point $P(0, 3)$ lies on C

(b) Find the equation of the tangent to the curve at P, giving your answer in the form $y = mx + c$, where m and c are constants to be found.

(2)

a) $f'(\alpha) = 0$ small angle approximation for $\cos\alpha$

$\therefore\ 2\alpha + \frac{1}{2}\left(1 - \frac{\alpha^2}{2}\right) = 0$ ①

$\times 2$

$4\alpha + 1 - \frac{\alpha^2}{2} = 0$

$\times -2$

$\alpha^2 - 8\alpha - 2 = 0$ ①

$\alpha = 8.243$ or $\alpha = -0.243$

choose $\alpha = -0.243$ as α is small. ①

b) $P(0,3)$

$f'(0) = \frac{1}{2}\cos 0 = \frac{1}{2}$ ← gradient (m) when $x = 0$.

$y - 3 = \frac{1}{2}(x - 0)$ ① ← uses $y - y_1 = m(x - x_1)$ with point P.

$y = \frac{1}{2}x + 3$ ①

7. **In this question you must show all stages of your working.**

Solutions relying on calculator technology are not acceptable.

A curve has equation

$$x^3 + 2xy + 3y^2 = 47$$

(a) Find $\frac{dy}{dx}$ in terms of x and y

(4)

The point $P(-2, 5)$ lies on the curve.

(b) Find the equation of the normal to the curve at P, giving your answer in the form $ax + by + c = 0$, where a, b and c are integers to be found.

(3)

a) $x^3 + 2xy + 3y^2 = 47$

$3x^2 + 2y + 2x\frac{dy}{dx} + 6y\frac{dy}{dx} = 0$ ① ①

$\frac{dy}{dx}(2x + 6y) = -3x^2 - 2y$ ①

$\frac{dy}{dx} = -\frac{3x^2 + 2y}{2x + 6y}$ ①

b) $P(-2, 5)$

$\left.\frac{dy}{dx}\right|_{x=-2,\, y=5} = -\frac{3(-2)^2 + 2(5)}{2(-2) + 6(5)} = -\frac{11}{13}$ ①

$\therefore\ m_{normal} = \frac{13}{11}$

$y - 5 = \frac{13}{11}(x + 2)$ ①

$11y - 55 = 13x + 26$

$13x - 11y + 81 = 0$ ①